APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

Scheme for Valuation/Answer Key

Scheme of evaluation (marks in brackets) and answers of problems/key

FIRST SEMESTER B.TECH DEGREE EXAMINATION(S), MAY 2019

Course Code: MA101

Course Name: CALCULUS

Max. Marks: 100 Duration: 3 Hours

PART A

1 a $\rho = \lim_{k\to\infty} \frac{3k-4}{4k-5} = \frac{3}{4} < 1$ (1)

Thus by Cauchy's Root test the series converges. (1)

b Series; f(0) =1,f'(0) = -1,f''(0)=2,f'''(0)=-6; (3)

$f(x) = 1-x+x^2- x^3$…………(1+1+1)… OR By Binomial series

2 a $z_{yy} = 48(3x-2y)^2, z_{yyy} = -192(3x-2y)$, (1)

$z_{yyyx} = -576$(Full marks may be given if the answer is correct to the question taken by student as there was lack of clarity in the power in the printed question paper) (1)

b $u_x = \frac{sec^2 x}{\tan x + tany + \tan z}$, $\sin 2x\ u_x = \frac{2\tan x}{\tan x + tany + \tan z}$ (1)

$\sin 2y\ u_y = \frac{2\tan y}{\tan x + tany + \tan z}$, $\sin 2z\ u_z = \frac{2\tan z}{\tan x + tany + \tan z}$ (1)

Substitution & getting $\sin 2x \frac{\partial w}{\partial x} + \sin 2y \frac{\partial w}{\partial y} + \sin 2z \frac{\partial w}{\partial y} = 2$ (1)

3 a $\left(\frac{d\vec{r}}{dt}\right) = -2\sin t\vec{i} + 2\cos t\,\vec{j} + \vec{k}$ ………(1) $\left\|\frac{d\vec{r}}{dt}\right\| = \sqrt{5}$ …………… (1) (2)

b $y = \int (\cos t\, i + \sin t\, j)\, dt$ ………………………………………..1 mark (1)

$y = \sin t\, i - \cos t\, j + \vec{c}$ ……………………………………………..1 mark (1)

Applying initial condition; $\vec{c} = i$ ………………………..1mark (1)

4 a $\int_0^1 \int_0^{x^2} 2\ dz\, dx$ ……………………………1 mark (2)

Ans:2/3 ……………………………1 mark

b $\iint xy\,dx\,dy = \int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\,dx\,dy$…………..(1) (3)

$= \int_0^b \frac{y}{2}\left(\frac{a^2}{b^2}(b^2-y^2)\right)dy$ ………………………..(1)

$= \left[\frac{a^2}{2b^2}\left(\frac{b^2y^2}{2}-\frac{y^4}{4}\right)\right]_0^b$……………………………..(1)

$= \frac{a^2b^2}{8}$

5 a $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 6xy^2$(1+1) (2)

OR Curl F=0(1+1)

b $\bar{r} = x\,\bar{\imath} + y\,\bar{\jmath} + z\,\bar{k}$, $r = \|\bar{r}\| = \sqrt{x^2+y^2+z^2}$ (3)

$\nabla.\frac{\bar{r}}{r^3} = \left(\frac{\partial}{\partial x}\bar{\imath} + \frac{\partial}{\partial y}\bar{\jmath} + \frac{\partial}{\partial z}\bar{k}\right).\left(\frac{x\,\bar{\imath}+y\,\bar{\jmath}+z\,\bar{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$ (1 mark)

$= \Sigma \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$ (1 mark)

= 0 (1 mark) (Full marks may be given with suitable step marks for alternate methods)

6 a *By stoke's theorem* $\int_C \bar{F}\,.\,d\bar{r} = \int_S \text{curl}\,\bar{F}.\,\bar{n}\,dS$ (1 mark) (2)

$\bar{F} = e^x\,\bar{\imath} + 2y\,\bar{\jmath} - \bar{k} \Rightarrow curl\,\bar{F} = 0$

Hence $\oint_C (e^x\,dx + 2y\,dy - dz) = 0$ (1 mark)

b *By Green's theorem*, $\int_C x\,dy - y\,dx = \iint_R \frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\,dA$ (1 mark) (3)

$= \iint_R 2\,dA$

= 2 x Area of circle (1)

=8 π (1 mark)

PART B

MODULE I

7 $$a_k = \frac{1}{(8k^2-3k)^{1/3}}, b_k = \frac{1}{(8k^2)^{1/3}} = \frac{1}{2k^{2/3}}$$ (1+1+1)

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = 1 > 0$$ (1)

(1)

By limit comparison test the series diverges.

(Full marks may be given if the answer is correct to the question taken by student as there was lack of clarity in the power in the printed question paper)

8 Let$\sum a_k = \sum_{k=0}^{\infty} \frac{(2x-1)^k}{3^{2k}}$. (1+1+1)

$l = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|2x-1|^{k+1} 3^{2k}}{3^{2k+2}|2x-1|^k} = \frac{|2x-1|}{9}$. (1)

By ratio test for absolute convergence, series converges absolutely only when $l < 1$. Therefore $|2x-1| < 9 \Rightarrow x \in (-4,5)$. (1)

At $x = -4$,$\sum a_k = \sum(-1)^k$, diverges. At $x = 5$,$\sum a_k = \sum 1^k$, diverges.

Interval of convergence $= (-4,5)$

Radius of convergence $= 9$.

9 $$\rho = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \frac{1}{e} < 1$$ (2+2) 1 (5)

Thus by Cauchy's Root test the series converges.

(Full marks may be given if the answer is correct to the question taken by student as there was lack of clarity in the power in the printed question paper)

MODULE II

10 $\frac{dw}{d\theta}$ formula and substitution (2)

$$\frac{dw}{d\theta} = \tan\theta \sec\theta$$ (2)

$$\frac{dw}{d\theta} \, at \; \theta = \frac{\pi}{4} = \sqrt{2}$$ (1)

OR Direct chain rule and substitution (suitable marks distribution)

11 $f_x = \frac{1}{x}, f_y = \frac{1}{y}$

$f_x(1,2) = 1, f_y(1,2) = \frac{1}{2}$ (1)

(1+1)

$L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = \ln 2 + x + \frac{y}{2} - 2$

$L(1.01,2.01) = 0.70814718, f(1.01,2.01) = 0.70808505$ (1)

Error $= L(1.01,2.01) - f(1.01,2.01) \approx 0.00006213$, Distance between P and $Q = 0.0141421356$ (1)

Error is less than $\frac{1}{250}$ times the distance between points P and Q.

12 $f_x - y - \frac{8}{x^2}, f_y - x - \frac{8}{y^2}, \quad \ldots\ldots\ldots\ldots\ldots (1)$ (5)

$$r = f_{xx} = \frac{16}{x^3}, s = f_{xy} = \frac{16}{y^3}, t = f_{yy} = 1$$ (1)

$f_x = 0, f_y = 0 \Rightarrow$ Critical point is $(2,2)$ (1)

$D = rt - s^2 = 3.$ (1)

At $(2,2), D > 0$ and $r > 0$. $(2,2)$is a relative minimum.(1)

MODULE III

13 $$T(t) = \frac{-a\sin t\vec{i} + a\cos t\vec{j} + c\vec{k}}{\sqrt{a^2 + c^2}}$$ .....(1+1) (5)

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = -\cos t\vec{i} - \sin t\vec{j}$$ .... (1+1+1)

14 $v(t) = -\cos t\, i + \sin t\, j + e^t k + \vec{c_1}$ ....(1) (5)

$r(t) = -\sin t\, i - \cos t\, j + e^t k + \vec{c_1} t + \vec{c_2}$ ... (1)

Applying initial conditions

$\vec{c_2} = -i + j$ .... (1)

$\vec{c_1} = i$ .... (1)

$r(t) = (t - \sin t - 1)\, i + (1 - \cos t)\, j + e^t k$ ......(1)

15 $F = z - x^2 - y^2$ (5)

$G = 3x^2 + 2y^2 + z^2 - 9$

$\nabla F = -2xi - 2yj + k$.........(1)

$\nabla G = 6xi + 4yj + 2zk$ ....(1)

$\nabla F(1,1,2) = -2i - 2j + k$, $\nabla G(1,1,2) = 6i + 4j + 4k$ ......(1)

$\nabla F \times \nabla G = -12i + 14j + 4k$ ...(1)

Equation of tangent line is$= \frac{x-1}{12} = \frac{y-1}{14} = \frac{z-2}{4}$ ....(1)

MODULE IV

16 Identification of Region ….(1) $\int_0^{a/\sqrt{2}} \int_0^{x} xy\,dy\,dx + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$ …..(2) (1) (2)

Ans = $\frac{a^3}{3\sqrt{2}}$ ……. (2) (2)

17 Identification of Region …… (1) (2)

$\int_0^2 \int_{y^2}^{6-y} xy\,dx\,dy$ ……(2) (3)

$\frac{1}{2}\int_0^2 (-y^5 + y^3 - 12y^2 + 36y)\,dy$ …………..(1)

Ans = 50/3 …..(1)

18 $\int_0^1 \int_{y^2}^1 x(1-x)\,dx\,dy$ ……(1) (2)

$\int_0^1 [\frac{x^2}{2} - \frac{x^3}{3}]_{y^2}^{1}$ ……………(1) (3)

$\int_0^1 \frac{1}{2}[1-y^4] - \frac{1}{3}[1-y^6]dy$ ……(1)

$[\frac{1}{2}(y-\frac{y^5}{5})]_0^1 - \frac{1}{3}[y-\frac{y^7}{7}]_0^1$ ……….(1)

$=\frac{4}{35}$ ……….(1)

MODULE V

19 $W = \int_C F.dr = \int_C (x^2+y^2)dx - xdy$ (2)

$$= \int_0^{\frac{\pi}{2}} -\sin\theta \ d\theta - \cos^2\theta \ d\theta = -\frac{\pi}{4} - 1$$ (3)

20 Curl F =0 (1)

$F = \nabla\emptyset$ (1)

$\frac{\partial\phi}{\partial x} = 6y^2, \frac{\partial\phi}{\partial y} = 12xy$ (1)

(2)

$\emptyset = 6xy^2 + k$

21 $div\ F - yz^2 + zx^2 + xy^2$ (2)

$curl\ F = (2xyz - yx^2)i - (zy^2 - 2xyz)j + (2xyz - xz^2)k$ (3)

22 $curl\,\bar{F} = \nabla \times \bar{F}$ (5)

$$= \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

=0 (1 mark)

Since $curl\,\bar{F}$ = 0 , the line integral is independent of the path.

(1 mark)

Consider any curve , say a line from (0,0,0) to (1,2,3),

$x = t, y = 2t, z = 3t$ (1 mark)

$\int_C (x^2 - yz)\bar{\imath} + (y^2 - zx)\bar{\jmath} + (z^2 - xy)\bar{k} \,.\, d\bar{r}$ =

$\int_C (x^2 - yz)\,dx + (y^2 - zx)dy + (z^2 - xy)dz = \int_0^1 18\,t^2\,dt$ =

6 (2 marks)

OR

Find Scalar potential $\varphi = \frac{x^3+y^3+z^3}{3} - xyz$............(2)

$$\int f.dr = [\varphi]_{(0,0,0)}^{(1,2,3)} \ldots\ldots(1)$$

$$= 6 \quad \ldots\ldots\ldots\ldots(1)$$

23 $\nabla^2 f(r) = \nabla.\nabla f(r)$ (1 mark) (1)

$$\nabla f(r) = \frac{\partial f(r)}{\partial x} i + \frac{\partial f(r)}{\partial y} j + \frac{\partial f(r)}{\partial z} k$$

$$= \frac{f'(r)}{r}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= \frac{f'(r)}{r}\bar{r}$$ **(1mark)** (1)

$$\nabla f'(r) = \frac{f''(r)}{r}\bar{r}$$

$$\nabla^2 f(r) = \nabla.\frac{f'(r)}{r}\bar{r} = f'(r)\nabla.\frac{\bar{r}}{r} + \nabla f'(r).\frac{\bar{r}}{r}$$ (1 mark) (1)

simplifying it gives

$$= \frac{2}{r} f'(r) + f''(r)$$ (2 marks) (2)

(Full marks may be given with suitable step marks for alternate methods)

MODULE VI

24 $\int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dy\,dx$ (1 mark) (1)

$$f = xy + y^2, g = x^2$$

$$\frac{\partial g}{\partial x} = 2x, \frac{\partial f}{\partial y} = x + 2y$$ (1 mark) (1)

$\int_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (x - 2y)\,dy\,dx$ (1 mark) (1)

$= \int_0^1 (x^{\frac{3}{2}} - x - x^3 + x^4)\,dx$ (1 mark) (1)

$= \frac{-3}{20}$ (1 mark) (1)

25 $\iint_\sigma f(x,y,z)\,dS = \iint_R f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$ dA (1 mark) (1)

$f(x,y,z) = z^2$, $z = g(x,y) = \sqrt{x^2 + y^2}$

$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$, $\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2}$ (1 mark) (1)

$\iint_\sigma z^2\,dS = \iint_R (x^2 + y^2)\,\sqrt{2}\,dA$ (1 mark) (1)

Putting,

$x = r\cos\theta, y = r\sin\theta, \quad dA = r\,dr\,d\theta, \quad 1 < r < 3, 0 < \theta < 2\pi$

$\iint_\sigma z^2\,dS = \iint_R (x^2 + y^2)\,\sqrt{2}\,dA = \sqrt{2}$

$\int_1^3 \int_0^{2\pi} r^2 r\,dr\,d\theta$

$= \sqrt{2}$

$\int_1^3 r^3\,dr \int_0^{2\pi} d\theta$

= $40\pi\sqrt{2}$ (2marks) (2)

(Full marks may be given if the answer is correct to the question taken by student as there was lack of clarity in the power in the printed question paper)

26 (i) $F(x,y,z) = (y+z)\bar{\imath} - xz\bar{\jmath} + x^2 siny\ \bar{k}$

$div\bar{F} = 0$. It has no sources or sinks (2 marks) (1+1)

(ii) $F(x,y,z) = x^3\bar{\imath} + y^3\bar{\jmath} + 2z^3\bar{k}$

$div\bar{F} = 3x^2 + 3y^2 + 3z^2 > 0$, for all points except at origin.

So it has sources at all points except at origin (2marks) (2)

Since $3x^2 + 3y^2 + 3z^2$ *cannot* be negative , it has no sinks. (1 mark) (1)

27 $div\ F = x$ (1+2+2)

$$\phi = \iint_{\sigma} F.n\ ds = \iiint_{G} div\ F\ dV = \iiint_{G} x\ dV = 3\int_0^2\int_0^{2-x}\int_0^{2-x-y} x\ dz\ dy\ dx = \frac{2}{3}$$

28 *By stoke's theorem* $\int_C \bar{F}\,.\,dV = \int\ \int_R$ curl $\bar{F}.\bar{n}$ dS (1 mark) (1)

$\bar{F} = xy\,\bar{\imath} + yz\,\bar{\jmath} + xz\,\bar{k}$

Curl $\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = -y\,\bar{\imath} - z\,\bar{\jmath} - x\,\bar{k}$ (1 mark) (1)

$x + y + z = 1 \;\; \Rightarrow z = 1 - x - y$

So $\bar{n} = -\frac{\partial z}{\partial x}\bar{\imath} - \frac{\partial z}{\partial y}\bar{\jmath} + \bar{k} = \bar{\imath} + \bar{\jmath} + \bar{k}$ (1 mark) (1)

The rectangular region in the xy plane is enclosed by $x + y - 1, x - 0, y - 0$

(1)

$\int \int_R \text{curl}\,\bar{F}.\bar{n}\,dS = \int \int_R (-y\,\bar{\imath} - z\,\bar{\jmath} - x\,\bar{k}).(\bar{\imath} + \bar{\jmath} + \bar{k})$ dA (1 mark)

$= \int \int_R \; -y \; -z \; -x$ dA

$= \iint_R -y - 1 + x + y - x \; dA$

$= \iint_R -1 \; dA = -\iint_R \; dA = -area\ of\ the\ triangle = -\frac{1}{2}$ (1)

(1 mark)

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